Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-10-20

Conceptual questions

Question 1. Consider the change of variables

$$x = u^2, y = 2v.$$

In each of the below cases, is it possible to rewrite the integral with this change of variables? If yes, do it. If not, explain why not.

(a) $\int_{-4}^{4} \int_{1}^{4} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$
(b) $\int_{1}^{4} \int_{-4}^{4} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$

The next two questions are rather difficult and subtle. They are beyond what you will actually need to know for using change of variables, so you are recommended to skip ahead to the computational problems below. In the following, T will denote a transformation from the uv-plane to the xy-plane.

Question 2. Let S be a region of the uv-plane that has area 4. Applying T to S results in a region R of the xy-plane (called the *image* of S under T; see exercises 15.9.7, 15.9.10 on your HW). In each of the following cases, is there enough information to determine the area of R, or does it depend on S? Potentially helpful reminder: the area of a region is just the integral of 1 over that region.

(a) The transformation T is given by

$$x = 2u + v, y = u + 2v.$$

(b) The transformation T is given by

$$x = e^u, y = v.$$

(c) (Hard) The transformation T is given by

$$x = \frac{v^2 - u^2}{u^2 + v^2}, y = \frac{2uv}{u^2 + v^2}.$$

Assume that the region R does not contain the origin $(u, v) = (0, 0)$.

(d) Actually I really meant to give you this transformation for the preceding part:

$$x = \frac{u^2 - v^2}{\sqrt{u^2 + v^2}}, y = \frac{2uv}{\sqrt{u^2 + v^2}}$$

Question 3. Let R be a region of the xy-plane that has area 4. The set S of all points in the uv-plane that are sent into R via T is called the *preimage* of R under T. That is,

$$S = \{(u, v) \in \mathbb{R}^2 : T(u, v) \in R\}.$$

In each of the same three cases as the preceding question, is there enough information to determine the area of S, or does it depend on R?

Computations

In a lot of exercises, the appropriate change of variables will be directly handed to you. But in a lot of cases, there is a natural choice.

Sometimes, the integrand is what gives away the appropriate change of variables.

Problem 1 (Stewart 15.9.25)**.** Evaluate the integral

$$\iint_R \cos\left(\frac{y - x}{y + x}\right) \mathrm{d}A$$

where R is the trapezoidal region with vertices $(1, 0), (2, 0), (0, 2), (0, 1)$.

In other cases, it might be the region of integration.

Problem 2 (Stewart 15.9.27)**.** Evaluate the integral

$$\iint_R e^{x+y}\, \mathrm{d}A$$

where R is given by the inequality $|x| + |y| \le 1$ (draw a picture).